Math 246A Lecture 30 Notes

Daniel Raban

December 7, 2018

1 Conclusion of Perron's Solution to the Dirichlet Problem

1.1 Limit of a maximal subharmonic function on the boundary of a domain

Let Ω be a bounded domain, and let f be real-valued on $\partial\Omega$ with $|f| \leq M$. We have that $V_f = \{v : v \text{ subharmonic in } \Omega, \limsup_{x \to \zeta} v(z) = f(\zeta)\ \forall \zeta \in \partial\Omega\}$.Let $u_f(z) = \sup\ [v(z) : v \in V_f\}$. Last time, we showed that u_f is harmonic.

Also recall the notion of regular points we introduced last lecture. A point $\zeta \in \partial\Omega$ is regular if there exists $w(z)$ which is continuous on $\overline{\Omega}$, harmonic on Ω, $w(z) > 0$ on $\overline{\Omega} \setminus \{\zeta\}$, and $w(\zeta) = 0$.

Theorem 1.1. *If ζ is regular and f is continuous at ζ, then* $\lim_{z \to \zeta} u_f(z) = f(\zeta)$.

Proof. Step 1: $\limsup_{z \to \zeta_0} u_f(z) \leq f(\zeta_0) + \varepsilon$. Take $\delta > 0$ such that $|\zeta - \zeta_0| < \delta$. There exists $\alpha > 0$ such that $w(z) - w(\alpha) > 0$ in $\Omega \setminus B(z_0, \delta)$, where w is from the definition of a regular point. Let $W(z) = f(\zeta_0) + \varepsilon + w(z)/\alpha(M0f(\zeta_0))$. W is continuous on $\overline{\Omega}$ and harmonic on Ω. If $\zeta \in B(\zeta_0, \delta)$, then $\limsup_{z \to \zeta_0} q(z) = f(\zeta) + \varepsilon \geq f(\zeta)$. If $\zeta \in \Omega \setminus B(\zeta_0, \delta)$, then $W(z) \geq \geq M - f(\zeta_0) + f(\zeta_0) + \varepsilon \geq M + \varepsilon$. Then for $v \in V_f$, $v \leq W$ by the maximum principle. Therefore, $u_f \leq W$. So $\limsup_{z \to \zeta_0} u_f(z) \leq f(\zeta_0) + \varepsilon$.

Step 2: $\liminf_{z \to \zeta_0} u_f(z) \geq f(\zeta_0) - \varepsilon$: Let $V = f(\zeta_0) - \varepsilon - w(z)/\alpha(M + f(\zeta_0))$. Then V is continuous on $\overline{\Omega}$ and is harmonic on Ω. If $\zeta \in B(\zeta_0, \delta)$, then $V(\zeta \leq f(\zeta_0 - \varepsilon \leq f(\zeta)$. If $\zeta \in \overline{\Omega} \setminus B(\zeta_0, \delta)$, then $V(\zeta) \leq -M - \varepsilon \leq f(\zeta)$. So if $v \in V_f$, $v \geq V$. Then $u_f \geq \liminf_{z \to \zeta_0} u_f(z) \geq \liminf_{z \to \zeta_0} V(z) \geq f(\zeta_0) - \varepsilon$. $\square$

Remark 1.1. The converse is true, as well. If this is true for all $f \in C(\partial\Omega)$, then ζ is regular.